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Stagnation points in flows about solid bodies *

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Abstract. A possible first step in determining the flow about a steadily advancing ship is to consider the ship plus its mirror image in the undisturbed free surface. If the ship has a bulbous bow, the question may be asked whether a stagnation point can be expected not only at the intersection of the stem and the undisturbed surface, but also at some point on the stem near the bulb. In analogous two-dimensional situations the latter could not happen. That it can happen in three dimensions is shown here by the example of two dipoles situated perpendicularly to an oncoming flow. Both two- and three dimensional versions are considered. The conditions under which these stagnation points can occur in three dimensions are determined and the reason why this does not happen in two dimensions is explained.

Keywords: stagnation points, potential flow, dipole pairs, vortex pairs.

1. Introduction

A rectilinear potential flow about a circle in the plane or about a sphere in three dimensions results in two stagnation points, one at each end of a diameter. For any bounded simply connected region in the plane it follows from Riemann's Mapping Theorem that there is an analytic function mapping the exterior of the unit circle into the exterior of the region and behaving like a rectilinear flow at infinity. Hence there are only two stagnation points on the boundary of the region in question. Although harmonic functions in three dimensions share many properties with analytic functions in the plane, there is no analogue of the Riemann theorem. In fact, there is only a very restricted set of transformations that preserve the property of being harmonic (see, *e.g.*, Kellogg [1, pp. 235–236]). It is natural to ask whether there can be more than two stagnation points in a potential flow about a bounded simply connected body in three dimensions. The question is raised in Kellogg [1, pp. 273–277] but not really answered. It is shown in Kellogg [1, p. 273] that there cannot be a continuous surface distribution of stagnation points (unless, of course, the potential function is constant). On the other hand, one knows that there can be continuous linear distributions of stagnation points if Laplace's equation can be separated in a particular coordinate system, as in $\Phi(x, y, z) = \varphi(x, y)Z(z)$ with $Z(z) = \text{const.}$ or $\Phi(r, \theta, z) = \varphi(r, z)\Theta(\theta)$ with $\Theta = \text{const.}$ One might be led to conjecture that any continuous line of stagnation points must be associated with a coordinate system in which Laplace's equation may be separated. The following potential function is

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a partial counterexample (JWK), for both the x-axis and the y-axis are lines of stagnation points:

$$\Phi(x, y, z) = \tfrac{1}{2}x^2y^2 - \tfrac{1}{2}(x^2 + y^2)z^2 + \tfrac{1}{6}z^4.$$

However, adding a uniform steady flow, say cx, does not lead to the flow about a body with a continuous curve of stagnation points.

As it is stated above, the problem of the existence of multiple isolated stagnation points appears to be one of only theoretical interest. Nevertheless, it turns out to be of some importance in the calculation of the wave resistance of ships. Proposals by Eiichi Baba [2], Charles Dawson [3], G. E. Gadd [4], and others for an improvement of Michell's [5] classical 'thin-ship' approximation all begin with the steady flow about the ship with the free-surface boundary condition replaced by that for a rigid flat surface. After reflection in this rigid boundary, one has the flow about a single body, the ship hull and its mirror image. If one is dealing with a ship with a bulbous bow, one asks oneself the question whether there can be a stagnation point on the bulb, and its reflection, as well as at the intersection of the rigid boundary and the stem, for the flow near the stem will be radically changed if a stagnation point does occur near the bulb. This question was the motivation for the investigation of the present paper.

A discussion by one of us (JVW) with Dawson in June 1978 concerning the possibilities of multiple stagnation points resulted in a letter from him dated 28 June 1978 describing his investigation of a three-dimensional body generated by two dipoles of equal moment situated on a line perpendicular to an oncoming steady rectilinear flow. As is well known, when the separation of the dipoles is zero, one streamline will generate a sphere with stagnation points at opposite ends of a diameter. Dawson correctly predicts the dipole separation at which each of the two stagnation points will begin to separate into three stagnation points, and also the (further) separation at which the single body will divide into two bodies. In addition, he computed the positions of the stagnation points lying on the central streamline as long as there is only one body.

Dawson died in January 1980 without having published any details concerning his calculations. In the present paper we shall present not only the analysis and computation necessary to substantiate Dawson's results, but also other relevant details accessible by exploiting modern computational capabilities such as *Mathematica*, which will be used below. In addition to the three-dimensional problem, we shall also treat the analogous two-dimensional problems for two dipoles and for two vortices. The analysis and the computation for these are simpler than for three dimensions, in particular, because of the presence of a stream function, but the results are relevant both for their similarities to and their differences from the three-dimensional case. These will be discussed at the end of the paper. We shall start with the two-dimensional results, first the vortices, then the dipoles, and finally we shall treat the three-dimensional case.

2. The velocity potentials

We shall first formulate the problem in three dimensions and then state the two-dimensional problems. The actual analysis will be developed in reverse order.

We suppose that there are two dipoles of moment μ and direction OX, one placed at $(0, a, 0)$ and the other at $(0, -a, 0)$, in a steady flow with velocity c in the direction OX. The velocity potential for the flow may then be written as follows:

$$\Phi(x, y, z) = x\left[c + \frac{\mu}{r_1^3} + \frac{\mu}{r_2^3}\right],$$
$$r_1^2 = x^2 + (y-a)^2 + z^2, \qquad r_2^2 = x^2 + (y+a)^2 + z^2. \tag{1}$$

The problem will be made dimensionless as follows:

$$(x, y, z) = \left(\frac{\mu}{c}\right)^{\frac{1}{3}} (\underline{x}, \underline{y}, \underline{z}), \qquad a = \left(\frac{\mu}{c}\right)^{\frac{1}{3}} \underline{a}, \qquad \Phi = \left(\frac{\mu}{c^2}\right) \underline{\Phi}. \tag{2}$$

Then, after dropping the underlining, we have the following equation for Φ

$$\Phi(x, y, z) = x\left[1 + \frac{1}{r_1^3} + \frac{1}{r_2^3}\right],$$
$$r_1^2 = x^2 + (y-a)^2 + z^2, \qquad r_2^2 = x^2 + (y+a)^2 + z^2. \tag{3}$$

There are other ways to make the equations dimensionless, but the present way has the virtue that the separation of dipoles is displayed, and this seems easier to intuit physically than, for example, having the dipoles at $(0, \pm 1, 0)$ with a variable dimensionless moment μ.

The analogous flow in the plane for a pair of similarly disposed dipoles can be formulated as an analytic function:

$$f(z) = cz + \frac{\mu}{z - ia} + \frac{\mu}{z + ia} = \phi(x, y) + i\psi(x, y). \tag{4}$$

Letting

$$z = \left(\frac{\mu}{c}\right)^{\frac{1}{2}} \underline{z}, \qquad a = \left(\frac{\mu}{c}\right)^{\frac{1}{2}} \underline{a}, \qquad f = (\mu c)^{\frac{1}{2}} \underline{f} \tag{5}$$

and again dropping the underlining for the dimensionless variables, we find

$$f(z) = z + \frac{1}{z - ia} + \frac{1}{z + ia} = \phi(x, y) + i\psi(x, y). \tag{6}$$

We shall also consider the flow in the plane about two vortices of strength $\pm\kappa$ at $z = \pm ia$ in a flow of velocity c in direction OX:

$$f(z) = cz + i\kappa \log \frac{z - ia}{z + ia} = \phi + i\psi. \tag{7}$$

In this case we let

$$z = \left(\frac{\kappa}{c}\right) \underline{z}, \qquad a = \left(\frac{\kappa}{c}\right) \underline{a}, \qquad f = \kappa \underline{f}, \tag{8}$$

resulting in

$$f(z) = z + i \log \frac{z - ia}{z + ia} = \phi + i\psi. \tag{9}$$

It is clear from the formulation of the problems that the streamlines will be symmetric with respect to both the (x, y) and the (y, z) planes for the three-dimensional problem. For the most part we shall restrict ourselves to showing traces of stream surfaces in the first quadrant of the (x, y) plane. For the two-dimensional problems the streamlines are symmetric with respect to the x and y axes, and we shall again restrict ourselves to showing results in only the first quadrant.

We shall ignore the behavior of the streamlines or surfaces emanating from the dipoles. These always lie within the closed streamline or surface that is the subject of this investigation. Similarly, the streamlines surrounding each of the two vortices and lying within the single large closed streamline will be ignored. It will be often convenient to refer to the single closed streamline or surface as the ‘body’, as though the interior of the closed streamline or surface had been replaced by solid material. In our figures we have shown only those streamlines exterior to the body that end in a stagnation point on the body.

3. Vortex pairs

The flow about a vortex pair has been treated by Milne–Thomson [6, pp. 344–345], but a figure is shown only for the case $a = \frac{1}{2}$. Since one of the aims of the present work is to examine the effect of separation upon the form of the generated body, and since the two-vortex flow is somewhat different from the two-dipole flow, we are including this case, following Milne–Thomson’s analysis.

From Equation (9) follows

$$f(z) = \phi + i\psi = x + iy + i\,\log\frac{r_1\,\exp(i\theta_1)}{r_2\,\exp(i\theta_2)} = x - (\theta_1 - \theta_2) + i\left(y + \log\frac{r_1}{r_2}\right), \qquad (10)$$

where $z - ia = r_1 \exp i\theta_1$ and $z + ia = r_2 \exp i\theta_2$. Thus

$$\psi(x, y) = y + \log\frac{r_1}{r_2} = y + \tfrac{1}{2}\,\log\frac{x^2 + (y-a)^2}{x^2 + (y+a)^2}. \qquad (11)$$

A streamline $\psi = \psi_0$ will then be represented by the equation

$$x^2 = \frac{(y+a)^2\,\exp 2(\psi_0 - y) - (y-a)^2}{1 - \exp 2(\psi_0 - y)}. \qquad (12)$$

The complex velocity is given by

$$f'(z) = u - iv = 1 + \frac{i}{z - ia} - \frac{i}{z + ia} = \frac{z^2 + a^2 - 2a}{z^2 + a^2}, \qquad (13)$$

so that a stagnation point is given by

$$z_0 = \pm\sqrt{a(2-a)}. \qquad (14)$$

We shall consider only the positive root since we are examining only the first quadrant. Evidently z_0 is real only if $0 < a < 2$ and $= 0$ if $a = 0$ or 2. It follows from (11) that $y = 0$ is a streamline and that this streamline will join with the ones determined by (12) with $\psi_0 = 0$ and

$0 < a < 2$. Figure 1 shows the first quadrant of several typical shapes for closed streamlines determined by (12) with $a = 0{\cdot}25$, $0{\cdot}50$, $1{\cdot}00$, $1{\cdot}50$, $1{\cdot}90$, and $2{\cdot}00$. If $a > 2$, the stagnation points determined by (14) no longer lie on the real axis but instead on the imaginary axis:

$$z_0 = \pm i\sqrt{a(a-2)}. \tag{15}$$

We can obtain the associated value of ψ_0 by substituting $x_0 = 0$ and $y_0 = \sqrt{a(a-2)}$ in (11). In Figure 1 two bodies have resulted when the two vortices are at a distance $2a = 4{\cdot}4$ apart. For this value of a, $x_0 = 0$, $y_0 = 0{\cdot}66$, and $\psi_0 = 0{\cdot}04096$.

Further derivatives of $f(z)$ lead to

$$\begin{aligned} f''(z) &= \frac{4az}{z^2 + a^2} = \phi_{xx} + i\psi_{xx}, \\ f'''(z) &= \frac{4a(a^2 - 3z^2)}{(z^2 + a^2)^3} = \phi_{xxx} + i\psi_{xxx} = -\phi_{xyy} + i\psi_{xxx}. \end{aligned} \tag{16}$$

and

$$f''(z_0) = \sqrt{\frac{2-a}{a}} = \phi_{xx}(x_0, 0) + i\psi_{xx}(x_0, 0),\; f'''(z_0) = \frac{2a-3}{a} = -\phi_{xyyy}(x_0, 0). \tag{17}$$

$\phi_{xx}(x_0, 0) = 0$ for $a = 2$, the boundary between the generation of one and two bodies. $\phi_{xyy}(x_0, 0) = 0$ for $a = \frac{3}{2}$, which marks the boundary between strictly convex bodies and bodies with a concave region near the stagnation point for $\frac{3}{2} < a < 2$. These same boundary markers will appear later for the flow about both the two-dimensional and the three-dimensional dipoles.

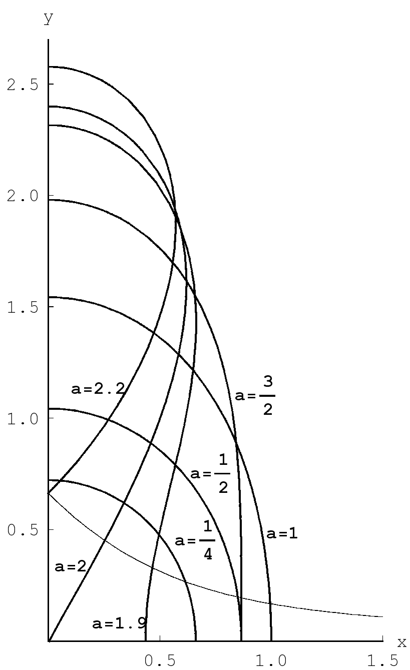


Figure 1. Two vortices.

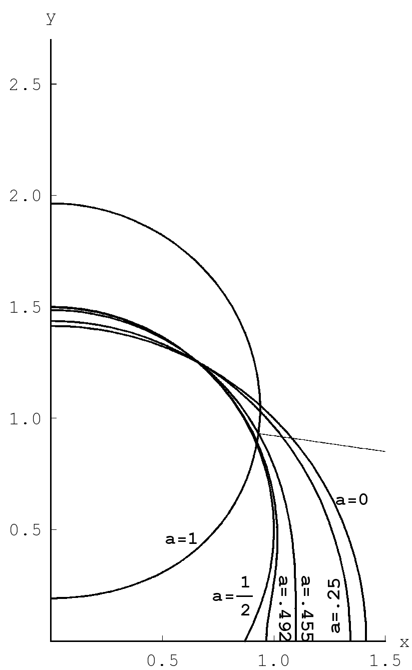


Figure 2. Two dipoles in the plane.

4. Dipole pairs in the plane

First we shall determine the positions of the stagnation points from

$$f'(z) = 1 - \frac{1}{(z-ia)^2} - \frac{1}{(z+ia)^2} = \frac{z^4 + 2(a^2-1)z^2 + a^4 + 2a^2}{(z^2+a^2)^2}. \tag{18}$$

Setting $f'(z) = 0$, we find

$$z_0^2 = 1 - a^2 \pm \sqrt{1-4a^2},$$

which is real and positive if $a \leqslant \frac{1}{2}$. The dipole separation $a = \frac{1}{2}$ evidently represents the upper bound of values of a for which a simple closed streamline is formed by the flow.

If $f(z)$ is separated into real and imaginary parts, $\phi + i\psi$, we find

$$\begin{aligned}\psi(x,y) &= y\left[1 + \frac{2(a^2 - x^2 - y^2)}{(x^2+y^2)^2 + a^2(2x^2 - 2y^2 + a^2)}\right]\\ &= y\left[\frac{(x^2+y^2-a^2-1)^2 + 4a^2x^2 - 1}{(x^2+y^2+a^2)^2 - 4a^2y^2}\right]\\ &= r\sin\theta\left[\frac{r^4 + 2r^2(a^2\cos 2\theta - 1) + 4a^2 + 2a^2}{r^4 + 2a^2r^2\cos 2\theta + a^2}\right]\end{aligned} \tag{19}$$

where, as usual, $x = r\cos\theta$ and $y = r\sin\theta$. As expected, $y = 0$ is a streamline for $\psi = 0$, and after some manipulation with the expression in [ ], we find that a further streamline is determined for $\psi = 0$ by

$$x^2 = 1 - a^2 - y^2 + \sqrt{1 + 4a^2(y^2-1)}, \tag{20}$$

or in polar coordinates by

$$r^2 = 1 - a^2\cos 2\theta \pm \sqrt{(1 - a^2\cos 2\theta)^2 - a^2(a^2+2)}. \tag{21}$$

As in the case of the vortex pairs, we shall not consider the streamline associated with the $-$ sign, which lies wholly within the closed one determined by the $+$ sign for $a \leqslant \frac{1}{2}$.

Figure 2 shows the part in the first quadrant of the streamline determined by (20) or (21) for $a = 0$, 0·25, 0·45509 $= [(2^{1/2}-1)/2]^{1/2}$, 0·49219 $= 63/128$, and 0·50. As for the case of the vortex pairs, we compute

$$f''(z) = \frac{4z(z^2-3a^2)}{(z^2+a^2)^3} \text{ and } f'''(z) = -\frac{12(z^4 - 6a^2z^2 + a^4)}{(z^2+a^2)^4},$$

$$f''(z_0) = \frac{4z_0(1 - 4a^2 + \sqrt{1-4a^2})}{(z_0^2+a^2)^3} = \phi_{xx} + i\psi_{xx}, \tag{22}$$

$$f'''(z_0) = -\frac{24[1 - 6a^2 + 4a^4 + (1-4a^2)^{\frac{3}{2}}]}{(z_0^2+a^2)^4} = -\phi_{xyy} + i\psi_{xxx}.$$

Evidently $f''(z_0) = 0$ for $a = \frac{1}{2}$, *i.e.* $\phi_{xx} = 0$ at the value of a separating one from two bodies. Substitution of $a = [(2^{1/2} - 1)/2]^{1/2}$ yields $\phi_{xyy} = 0$, although this is not so obvious. As with the vortex pairs, $\phi_{xyy} = 0$ marks the boundary between strictly convex bodies and ones with a concave region near the stagnation point.

Note especially, in comparing Figure 2 and Figure 1, the difference between the bodies with the values of a approaching the largest allowable value before two bodies are formed. For the vortex-pair bodies the neck steadily diminishes until it finally vanishes at $a = 2$, whereas the 'last' dipole-pair body still has a substantial neck joining the two halves. Also, in both cases the streamlines intersect the x-axis normally for $a <$the critical value, whereas *at* the critical value the streamline makes an angle of 60° with the x-axis. Note also that not only does the value $a = 0$ give a circle of radius $2^{1/2}$, but the body at the limiting value, $a = \frac{1}{2}$, consists of two circles with centers at $(0, \pm\frac{1}{2})$ and radii 1.

We still haven't considered how we shall deal with separations $a > \frac{1}{2}$. Now z_0^2 becomes complex with both real and imaginary parts. By substituting (x_0, y_0) for (x, y) in the expansion for $\psi(x, y)$ in (19), we may determine the value of $\psi(x_0, y_0) = \psi_0$ that is associated with the stagnation point. Unfortunately, the solution of the resulting equation for the streamline is no longer simple. As inspection of (19) will show, it is still possible to formulate it as a quadratic equation in x^2, but with very complicated expressions in y, a, and ψ_0 for the coefficient of x^2 and for the constant term, but still polynomial forms. The polar equation can be manipulated into the following cubic equation in $Y = r \sin\theta$:

$$4r^2Y^3 - 4\psi_0 Y^2 - (r^4 + 3)Y + \psi_0(r^2 + 1)^2 = 0 \tag{23}$$

in which one then assigns values to r and solves for Y. The same equation can also provide a fifth-degree polynomial in r to be solved (numerically) for assigned values of θ.

Figure 2 shows the first-quadrant part of the closed streamline (one of two) formed when the dipole separation is determined by $a = 1$. In this case $z_0 = (3/4)^{1/4}(1+i)$. The associated value of ψ_0 is $(3/4)^{1/4}(3 - 3^{1/2})/2 = 0{\cdot}58998$. The dramatic difference in the shapes of the 'last' single bodies for the two-vortex and the two-dipole bodies is repeated here for the separated bodies.

5. Dipole pairs in three dimensions

The analysis of the irrotational motion about two dipoles is noticeably more difficult than for the analogous problem in the plane. This is certainly a result of the absence of a stream function, which allowed one to derive immediately an equation describing the streamlines. Nevertheless, some conclusions can be drawn before it becomes necessary to have recourse to sophisticated computing methods.

We begin with Equation (3) and compute some derivatives:

$$\begin{aligned}
\Phi_x &= 1 + \frac{1}{r_1^3} + \frac{1}{r_2^3} - 3x^2\left(\frac{1}{r_1^5} + \frac{1}{r_2^5}\right),\\
\Phi_y &= -3x\left(\frac{y-a}{r_1^5} + \frac{y+a}{r_2^5}\right), \qquad \Phi_z = -3xz\left(\frac{1}{r_1^5} + \frac{1}{r_2^5}\right),\\
\Phi_{xx} &= -9x\left(\frac{1}{r_1^5} + \frac{1}{r_2^5}\right) + 15x^3\left(\frac{1}{r_1^7} + \frac{1}{r_2^7}\right),
\end{aligned} \tag{24}$$

$$\Phi_{yy} = -3x\left(\frac{1}{r_1^5}+\frac{1}{r_2^5}\right) + 15x\left(\frac{(y-a)^2}{r_1^7}+\frac{(y+a)^2}{r_2^7}\right),$$

$$\Phi_{zz} = -3x\left(\frac{1}{r_1^5}+\frac{1}{r_2^5}\right) + 15xz^2\left(\frac{1}{r_1^7}+\frac{1}{r_2^7}\right),$$

$$\Phi_{xyy} = -3\left(\frac{1}{r_1^5}+\frac{1}{r_2^5}\right) + 15x^2\left(\frac{1}{r_1^7}+\frac{1}{r_2^7}\right) + 15\left[\frac{(y-a)^2}{r_1^7}+\frac{(y+a)^2}{r_2^7}\right]$$
$$-105x^2\left[\frac{(y-a)^2}{r_1^9}+\frac{(y+a)^2}{r_2^9}\right].$$

The definition of a stagnation point requires $\Phi_x(x_0, y_0, z_0) = \Phi_y = \Phi_z = 0$, which in turn requires $z_0 = 0$ if $x_0 \neq 0$. Hence, from now on we may drop the z^2 in the definitions of r_1 and r_2 and examine the behavior of stagnation points only in the (x, y)-plane. We note also that $\Phi_y = 0$ if $y = 0$. Moreover, if $a = 0$, then $\Phi_y = 0$ and $x \neq 0$ require that $y = 0$. Since, in addition, $\partial\Phi_y/\partial a = 0$ for $a = 0$, it follows, at least for sufficiently small a, that $\Phi_y = 0$ implies that $y = 0$. Hence, for sufficiently small values of a we shall assume that at a stagnation point both $y = 0$ and $z = 0$ hold and concentrate on the implications of $\Phi_x(x, 0, 0) = 0$, which we now examine:

$$\Phi_x(x, 0, 0) = 1 + \frac{2}{(x^2+a^2)^{3/2}} - \frac{6x^2}{(x^2+a^2)^{5/2}} = 1 + \frac{2(a^2-2x^2)}{(x^2-a^2)^{5/2}}. \tag{25}$$

Then $\Phi_x(x_0, 0, 0) = 0$ yields

$$(x_0^2 + a^2)^{5/2} = 2(2x_0^2 - a^2), \tag{26}$$

and with $\xi = x_0^2/a^2$,

$$a^3 = \frac{2(2\xi-1)}{(\xi+1)^{5/2}}. \tag{27}$$

The following equations will be useful:

$$\Phi_{xx}(x, 0, 0) = \frac{6x(2x^2-3a^2)}{(x^2+a^2)^{7/2}},$$

$$\Phi_{xx}(x_0, 0, 0) = \frac{3x_0(2x_0^2-3a^2)}{(2x_0^2-a^2)(x_0^2+a^2)} = \frac{3\xi^{1/2}(2\xi-3)}{(2\xi-1)(\xi+1)a},$$

$$\Phi_{yy}(x, 0, 0) = \frac{6x(4a^2-x^2)}{(x^2+a^2)^{7/2}},$$

$$\Phi_{yy}(x_0, 0, 0) = \frac{3x_0(4a^2-x_0^2)}{(2x_0^2-a^2)(x_0^2+a^2)} = \frac{3\xi^{1/2}(4-\xi)}{(2\xi-1)(\xi+1)a},$$

$$\phi_{zz}(x_0, 0, 0) = \frac{-3x_0}{(2x_0^2 - a^2)} < 0 \text{ for } x_0 > 0, \tag{28}$$

$$\Phi_{xyy}(x_0, 0, 0) = \frac{3(4x_0^4 - 27a^2x_0^2 + 4a^2)}{(2x_0^2 - a^2)(x_0^2 + a^2)^2} = \frac{3(4\xi^2 - 27\xi + 4)}{(2\xi - 1)(\xi + 1)^2 a^2},$$

$$\Phi_{xy}(x, 0, 0) = \Phi_{xyy}(x, 0, 0) = \Phi_{yyy}(x, 0, 0) = 0.$$

Figure 3 shows the relation between a and x_0, the position of the stagnation point associated with a. Following our usual procedure of disregarding the streamlines and their stagnation points that lie inside the 'body', we shall consider only that part of the curve to the right of the maximum, which occurs at $a = [4(2/5)^{5/2}]^{1/3} = 0{\cdot}73972$, $x_0 = 0{\cdot}90597$, corresponding to $\xi = 3/2$. This is also the value of a for which $\Phi_{xx}(x_0, 0, 0) = 0$ and is the largest value of a before the single closed streamline divides into two closed streamlines. $a = 0$ corresponds to $x_0 = 4^{1/3} = 1{\cdot}58740$, the radius of the sphere generated by a single dipole. $\Phi_{xx}(x_0, 0, 0) > 0$ in this interval. $\Phi_{yy}(x_0, 0, 0) = 0$ for $a = 0{\cdot}63033$ and is < 0 for smaller values of a. $\Phi_{xxx}(x_0, 0, 0) = 0$ for $a = 0{\cdot}68528$ and is < 0 for smaller values. $\Phi_{xyy}(x_0, 0, 0) > 0$ for $a < 0{\cdot}53517$, where it $= 0$. The usefulness of these values will presently become evident.

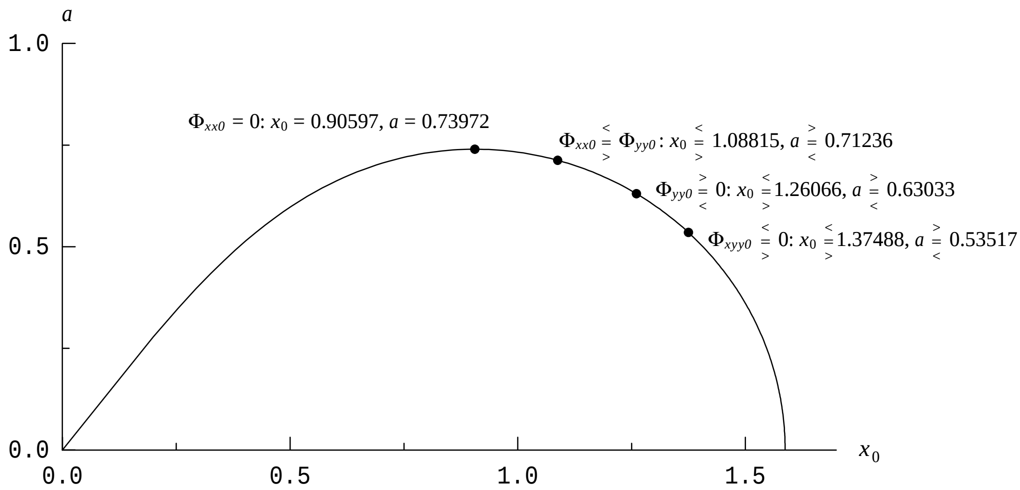


Figure 3. Relation between a and x_0 in three dimensions.

In the two cases of planar motion, determination of the streamlines, and, in particular, of the closed streamlines, was relatively straightforward. One had only to assign an appropriate value to the stream function (in fact, 0 for a single closed streamline) and then solve a fairly uncomplicated equation for the streamline. Also, one knew that there were only two stagnation points. In the present case, even though we are restricting ourselves for the moment to the (x, y)-plane, there is no stream function and we are not sure that there are only two stagnation points except when a is sufficiently small.

What is now necessary is to solve a set of three differential equations to determine a streamline, $\mathbf{r}(t) = (x(t), y(t), z(t))$, through a given point (x_1, y_1, z_1):

$$\frac{d\mathbf{r}}{dt} = \nabla\Phi, \quad \mathbf{r}(t_1) = \mathbf{r}_1, \tag{29a}$$

or

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\nabla\Phi}{|\nabla\Phi|}, \quad \mathbf{r}(s_1) = \mathbf{r}_1. \tag{29b}$$

The situation of most interest to us is, of course, when the initial condition is the stagnation point $\mathbf{r}_0 = (x_0, 0, 0)$.

Before proceeding further, we call attention to an interesting property of the stream surface passing through the stagnation point at $(x_0, 0, 0)$. If we write Equation (3) in cylindrical coordinates about the y-axis, with $x = r \cos\theta$ and $z = r \sin\theta$, we find

$$\Phi(r, \theta, y) = \cos\theta\varphi(r, y), \tag{30a}$$

where

$$\varphi = r\left[1 + \frac{1}{\{r^2 + (y-a)^2\}^{3/2}} + \frac{1}{\{r^2 + (y+a)^2\}^{3/2}}\right]. \tag{30b}$$

The stagnation point $(x_0, 0, 0)$ is now identified by $(r_0, 0, 0)$ with $r_0 = x_0$. Equations (29a) then become

$$r'(t) = \cos\theta\varphi_r(r, y), \theta'(t) = \sin\theta\varphi(r, y)/r^2, y'(t) = \cos\theta\varphi_y(r, y). \tag{30c}$$

From the first and third equations follows

$$\frac{\mathrm{d}r}{\mathrm{d}y} = \frac{\varphi_r(r, y)}{\varphi_y(r, y)}, \quad r(0) = r_0, \tag{30d}$$

which indicates that the stream surface is symmetric about the y-axis. In order to find the streamlines themselves, we must solve the second and third equations of (30c) with $r = r(y(t)), r(y)$ having already been determined by (30d). We are evidently losing no information by restricting our attention to the (x, y)-plane.

We shall start by examining the behavior of the streamlines near the stagnation point $(x_0, 0, 0)$ for a given value of a. If we expand $\Phi(x, y, 0)$ in a power series about $(x_0, 0, 0)$, discarding terms that we know to be 0, we find for the first few terms the following:

$$\Phi(x, y, 0) - \Phi(x_0, 0, 0) = \tfrac{1}{2}\Phi_{xx0}(x - x_0)^2 + \tfrac{1}{2}\Phi_{yy0}y^2 + \tag{31}$$

$$\tfrac{1}{6}\Phi_{xxx0}(x - x_0)^3 + \tfrac{1}{2}\Phi_{xyy0}(x - x_0)y^2 + \dots,$$

where we have written Φ_{xx0} for $\Phi_{xx}(x_0, 0, 0)$, etc. We may replace Equations (29) by the simpler single equation

$$\Phi_x(x, y, 0)\,\mathrm{d}y - \Phi_y(x, y, 0)\,\mathrm{d}x = 0, \tag{32}$$

which is a direct expression of the orthogonality of the streamlines and the equipotential lines in the (x, y)-plane.

The values of a given above for the vanishing of Φ_{xx0}, Φ_{yy0}, and Φ_{xyy0} already indicate that three-dimensional behavior is going to be different from that in two dimensions. In two dimensions Φ_{xx} and Φ_{yy} vanish together. The fact that in three dimensions $\Phi_{yy} < 0$ for small

a and $\Phi_{xx} > 0$ indicates that the equipotential curves in this region will be hyperbolas in the lowest approximation, whereas in the interval where $\Phi_{yy} > 0$ they will be ellipses. From Equation (31) the expansions for Φ_x and Φ_y in the neighborhood of $(x_0, 0, 0)$ begin as follows:

$$\Phi_x(x, y, 0) = \Phi_{xx0}(x - x_0) + \tfrac{1}{2}\Phi_{xxx0}(x - x_0)^2 + \tfrac{1}{2}\phi_{xyy0}y^2 + \dots, \tag{33}$$

$$\Phi_y(x, y, 0) = \Phi_{yy0}y + \Phi_{xyy0}(x - x_0)y + \dots.$$

In the very lowest order (32) becomes

$$\Phi_{xx0}(x - x_0)\,\mathrm{d}y - \Phi_{yy0}y\,\mathrm{d}x = 0, \tag{34}$$

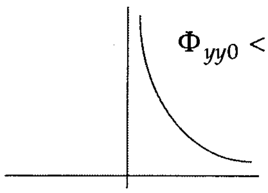
with the obvious solution

$$x - x_0 = Cy^p, \quad \text{where } p = \Phi_{xx0}/\Phi_{yy0}. \tag{35}$$

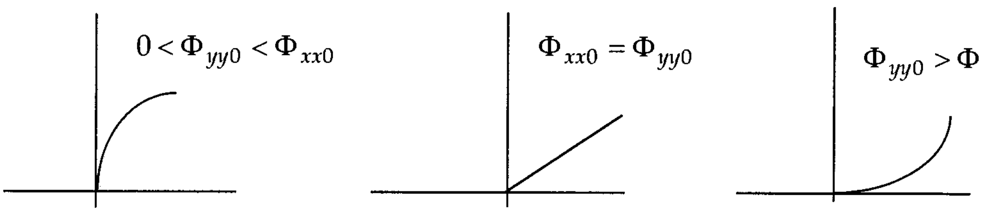
It is evident that in the neighborhood of $(x_0, 0, 0)$ the behavior of (35) changes radically depending upon the value of Φ_{yy0}. In particular, the significant regions, or values, are

$$\Phi_{yy0} < 0,\ \Phi_{yy0} = 0,\ 0 < \Phi_{yy0} < \Phi_{xx0},\ \Phi_{yy0} = \Phi_{xx0},\ \Phi_{yy0} > \Phi_{xx0},\ \Phi_{xx0} = 0.$$

The values $\Phi_{yy0} = 0$ and $\Phi_{xx0} = 0$ do not yield much useful information at this level of approximation, but serve as boundaries. In the region $\Phi_{yy0} < 0$ the solution with $x = x_0$, $y = 0$ as initial value is the y-axis. However, a streamline through a nearby point, say $(x_0 + \varepsilon, \eta)$ will have approximately the following shape:



In the region $\Phi_{yy0} > 0$ the behavior of streamlines in the first quadrant in the immediate neighborhood of $(x_0, 0)$ will be as shown below:



More detailed local information is obtained by retaining the quadratic terms in (33). Except for the separation associated with $\Phi_{xyy0} = 0$, the resulting differential equations do not allow a solution in closed form (as far as we could determine). To provide this information we fell back upon numerical solutions provided by *Mathematica*. Figure 4 shows the local behavior for several values of a that illustrate the regions mentioned above, plus one significant value not determined by the linear terms.

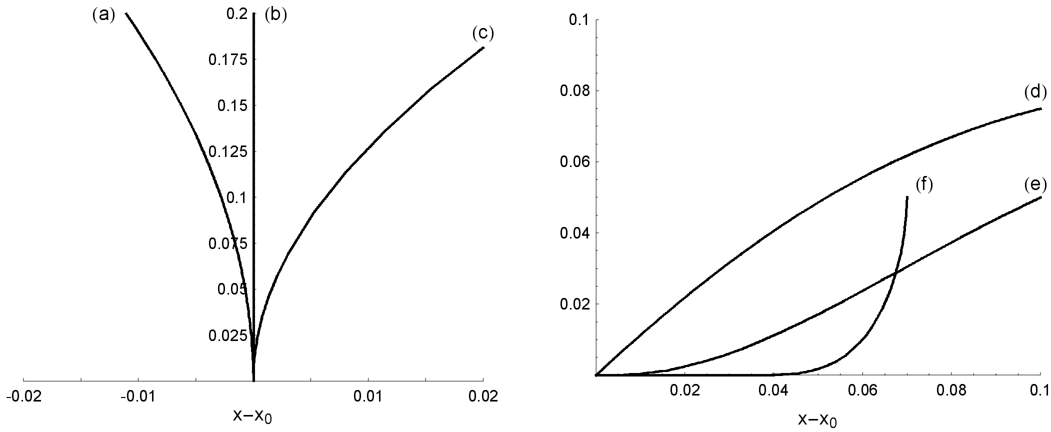


Figure 4. Behavior of streamlines near x_0 for various of a. (a): $a = 0{\cdot}25$, $\Phi_{xyy0} > 0$, (b): $a = 0{\cdot}53517$, $\Phi_{xyy0} = 0$, (c): $a = 0{\cdot}63063$, $\Phi_{yy0} > 0$, (d): $a = 0{\cdot}71236$, $\Phi_{xx0} > 0 = \Phi_{yy0}$, (e): $a = 0{\cdot}732$, $\Phi_{xx0} < \Phi_{yy0}$, (f): $a = 0{\cdot}739$, $\Phi_{xx0} = 0$.

As in the two-dimensional cases, the value of a associated with $\Phi_{xyy0} = 0$, namely, $a = 0{\cdot}53517$, separates the interval $0 \leqslant a \leqslant 0{\cdot}53517$ where the closed streamline is strictly convex from the interval $0{\cdot}53517 < a < 0{\cdot}71236$ where the closed streamline is concave near the stagnation point ($a = 0{\cdot}71236$ is the value of a at which $\Phi_{xx0} = \Phi_{yy0}$). In order to confirm this, consider the differential equation (32) with the approximations for Φ_x and Φ_y shown in (33). If we set $x = x_0$, the slopes along the line $x = x_0$ are given by $\frac{\mathrm{d}x}{\mathrm{d}y} = (\frac{\Phi_{xyy0}}{2\Phi_{yy0}})y$, and evidently, if $\Phi_{xyy0} = 0$, $x = x_0$ is a solution of the differential equation (32, 33) with a singular point at $(x_0, 0)$. Thus, in the immediate neighborhood of $(x_0, 0)$, where the approximation (33) holds, the streamline has no curvature. However, this does not indicate the curvature for separations a on either side of $a = 0{\cdot}53517$ where $\Phi_{xyyo} = 0$. For this we return to the differential equation (32, 33):

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{1}{\Phi_{yy0}y}[\Phi_{xx0}(x - x_0) + \tfrac{1}{2}\Phi_{xxx0}(x - x_0)^2 + \tfrac{1}{2}\Phi_{xyy0}y^2]\left[1 + \frac{\Phi_{xyy0}}{\Phi_{yy0}}x\right]^{-1} \\ &= \frac{\Phi_{xx0}}{\Phi_{yy0}}\frac{x}{y} + \frac{\Phi_{xyy0}}{2\Phi_{yy0}}y - \frac{\Phi^2_{xyy0}}{2\Phi^2_{yy0}}xy + O(x^2). \end{aligned} \tag{36}$$

Substitution of $x = u(y)y^p$, $p = \frac{\Phi_{xx0}}{\Phi_{yy0}}$ and neglect of terms of $O(x)$ ultimately yields the solution

$$x = \frac{1}{2}\frac{\Phi_{xyy0}}{2\Phi_{yy0} - \Phi_{xx0}}y^2 + Cy^p, \quad \text{if } 2\Phi_{yy0} - \Phi_{xx0} \neq 0. \tag{37}$$

For $\Phi_{yyo} < 0 (a < 0{\cdot}63033)$, the condition $x(0) = 0$ implies that $C = 0$ and hence that $x''(0) = \Phi_{xyyo}/(2\Phi_{yyo} - \Phi_{xxo})$. For the interval where $\Phi_{xyyo} > 0 (a < 0{\cdot}53517)$ evidently $x'' < 0$, and where it is > 0, $x'' > 0$. When $\Phi_{yyo} > 0 (a > 0{\cdot}63033)$, the condition $x(0) = 0$ leaves C undetermined, and it must be determined by the position of the neighboring stagnation point, which we know exists since $\Phi_{yyo} > 0$ (see (38) below). From the geometry of the stream surface it is plausible to assume that $C > 0$. Now x'' is given by the equation

$$x'' = \frac{\Phi_{xyy0}}{2\Phi_{yy0} - \Phi_{xx0}} + C\frac{\Phi_{xx0}}{\Phi_{yy0}}\frac{\Phi_{xx0} - \Phi_{yy0}}{\Phi_{yy0}}y^{p-2}.$$

Evidently $x'' > 0$ as long as $\Phi_{xxo} > 2\Phi_{yyo}(a < 0{\cdot}69139)$. Because of the predominance of the second term in x'' near $y = 0$, x'' remains > 0 in the interval between $\Phi_{xxo} = 2\Phi_{yyo}(a = 0{\cdot}69139)$ and $\Phi_{xxo} = \Phi_{yyo}(a = 0{\cdot}71236)$. For $a > 0{\cdot}71236$, $x'' < 0$ for $y > 0$. These predictions are all borne out by the numerical solutions shown in Figure 5.

The value $a = 0{\cdot}63033$ associated with $\Phi_{yy0} = 0$ marks another important transition. For $a > 0{\cdot}63033$ there is an additional stagnation point in the first quadrant of the (x, y)-plane, one with $y > 0$. The value $a = 0{\cdot}71236$ marks the advent of a new phenomenon, the appearance of a discontinuity in the slope, a corner that occurs not only in the trace in the (x, y)-plane, but, of course, in all traces with $\theta = \text{const}$. For values of $a > 0{\cdot}71236$ this corner becomes an inward-pointing cusp. For $a = 0{\cdot}73972$, when $\Phi_{xx0} = 0$, this cusp becomes exponentially sharp, not just the power-law cusp of Equation (35). As noted earlier, for separations $a > 0{\cdot}73972$ two bodies are formed.

Figure 5 shows the full closed streamline in the first quadrant for the same typical values of a as in Figure 4. These also were obtained by means of the *Mathematica* program, but for the numerical solution of the complete differential equations. Figure 5 shows four cases where the second stagnation point has appeared, together with the streamline issuing from it.

Dawson did not have *Mathematica* available, so that it seems appropriate to ask whether from (33) one could deduce the fact that there is only one stagnation point for $a \leqslant 0{\cdot}63033$, where $\Phi_{yy0} = 0$, but two or more such points for $a > 0{\cdot}63033$. In fact, if we set $\Phi_x = \Phi_y = 0$ in (33) and treat them as a pair of equations for determining the coordinates of a further stagnation point (or points), we find

$$x = x_0 - \frac{\Phi_{yy0}}{\Phi_{xyy0}}, \qquad y^2 = \frac{\Phi_{yy0}}{\Phi^2_{xyy0}}\frac{2\Phi_{xx0} - \Phi_{yy0}\Phi_{xxx0}}{\Phi_{xyy0}}. \tag{38}$$

If $a < 0{\cdot}63033$, this yields $y^2 < 0$ (even though $\Phi_{yy0}\Phi_{xxx0} > 0$ for small a, it is still overbalanced by $2\Phi_{xx0}$), and hence no real solution for y. For $a > 0{\cdot}63033$, however, $y^2 > 0$, and hence there exists a pair of nonzero real solutions. Note that for $a > 0{\cdot}63033$, $\Phi_{yy} > 0$ and $\Phi_{xyy} < 0$, so that the solution is somewhat to the right of x_0, as we expect. When $a = 0{\cdot}63033$, the two solutions coalesce with the stagnation point at $(x_0, 0)$.

Finally, Figures 6a,b,c show, again via *Mathematica*, three-dimensional illustrations of the first-octant stream surface together with streamlines and equipotential lines lying on the surface. Note that in Figures 6b and 6c the behavior in the neighborhood of the second stagnation point is clearly displayed. Figure 6a shows the stream surface for the largest value of a for which there is a single stagnation point in the first octant.

6. Some final remarks

The qualitative difference between two and three dimensions that has been noted above is chiefly a result of the following facts. In two dimensions Φ_{xx0} and Φ_{yy0} vanish together at the two stagnation points associated with the largest separation a before a single closed

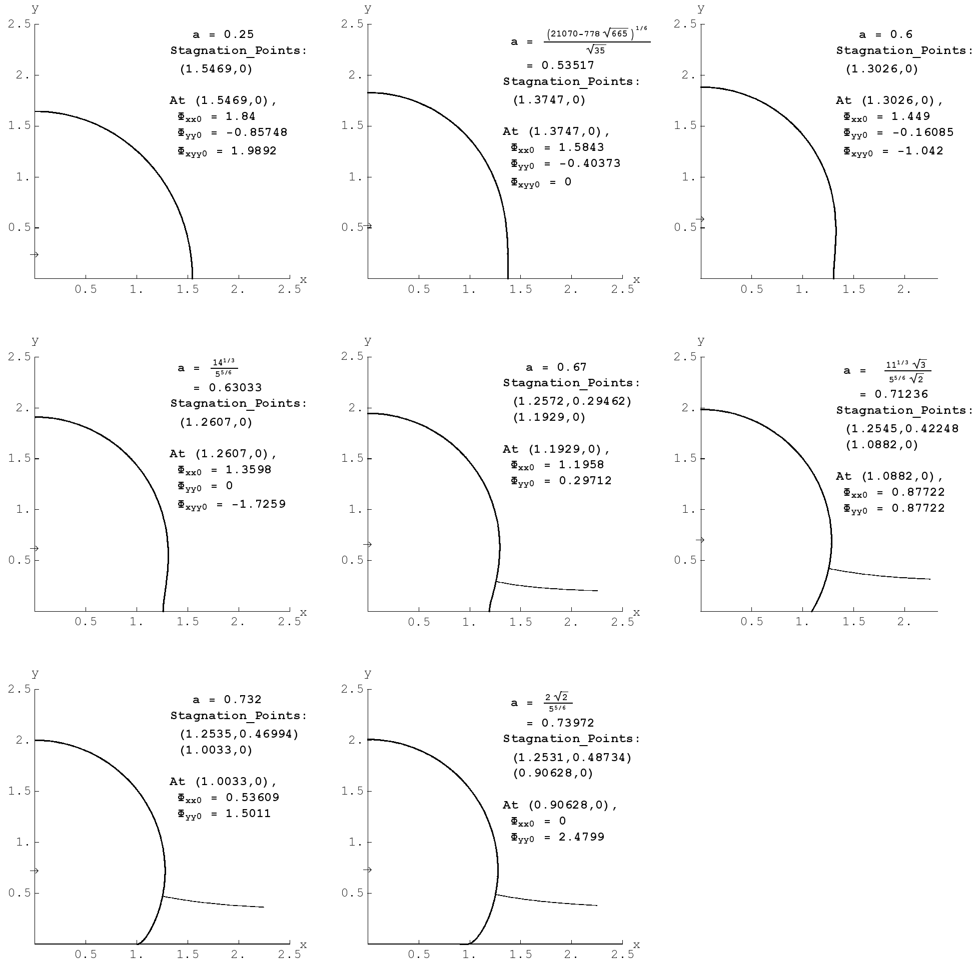


Figure 5. Traces of streams surfaces in the first quadrant of (x, y)-plane for values of a with only one stagnation point ($a \leqslant 0{\cdot}63033$), and with two such points ($0{\cdot}63033 < a \leqslant 0{\cdot}73972$).

stream body splits into two bodies, with, of course, two stagnation points on each. In three dimensions, however, $\Phi_{yy0} = 0$ at a stagnation point associated with a *smaller* value of a than that at which $\Phi_{xx0} = 0$, which again occurs at the largest value of a before the single closed stream body divides into two bodies. However, the existence of an interval of a for which $\Phi_{yy0} > 0$ has as a consequence the presence of two further stagnation points with $y \neq 0$. Thus there exists an interval of dipole separations for which there is only one closed stream body, but three stagnation points on each side. Furthermore, there exists an interval of separations for which $\Phi_{yy0} \geqslant \Phi_{xx0} > 0$, and this implies that the single body is not smooth at the waist, *i.e.* at the intersection of the stream body with the plane perpendicular to and bisecting the line joining the two dipoles. In two dimensions this nonsmooth behavior can occur only at the 'last' single body when $\Phi_{xx0} = \Phi_{yy0} = 0$, the only separation at which the two are equal.

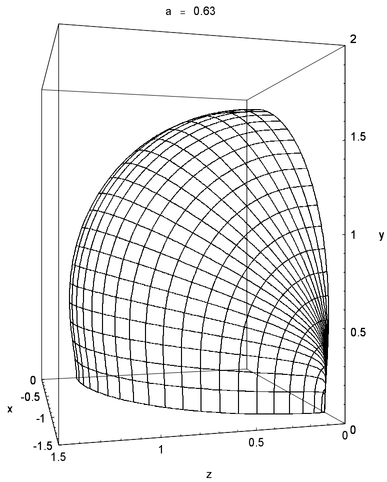


Figure 6a. 3-dimensional view of stream surface in the first octant for $a = 0{\cdot}63033$, the largest value of a for which there is a single stagnation point ($\Phi_{yy0} = 0$). Note that there, and in Figure 6b and 6c, streamlines and equipotential lines are shown.

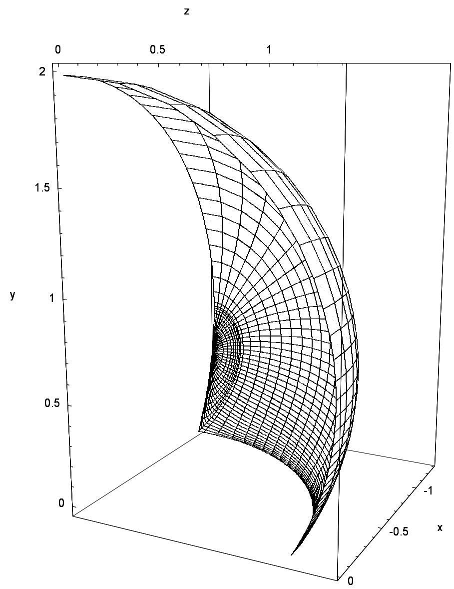


Figure 6b. 3-dimensional view of stream surface in the first octant for $a = 0{\cdot}71236$ where $\Phi_{xx0} = \Phi_{yy0}$.

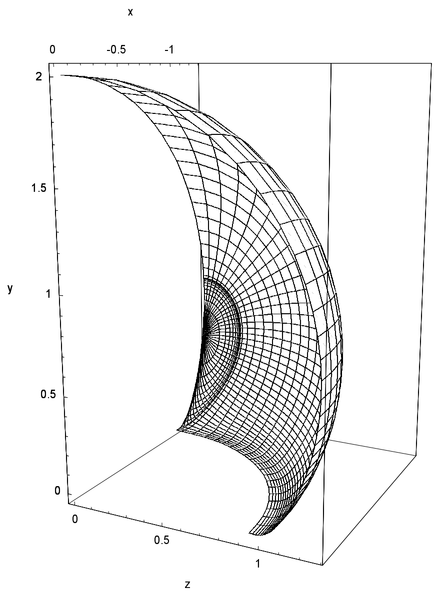


Figure 6c. 3-dimensional view of stream surface in the first octant for $a = 0{\cdot}73972$ where $\Phi_{xx0} = 0$, the largest value of a before two bodies are formed.

There remains a question of procedure. Since we could have produced the three-dimensional figures of Figure 6 at the very beginning, one may ask if there has been any point in presenting the more detailed analysis that has preceded it? The original question concerning the occurrence of multiple stagnation points on closed bodies could have been answered immediately. The two cases of planar motion, although perhaps not uninteresting, in the light of the preceding paragraph, could have been omitted as unnecessary if only the question of multiple stagnation points is of interest. The analysis leading up to Figure 4 could also have been omitted. Some numerical experimenting with the full equations used to produce Figure 5 would presumably have revealed the same changes in behavior as a increased, even if not with the same precision. It is evident that having available a tool like *Mathematica* can change the strategy for approaching a problem where difficult computations are involved. Whether it alone can produce sufficient and desired insight may be debated. Perhaps we should have started with Figure 6.

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